External trade

- **Example.** Consider three countries 1, 2, and 3. In each country, consumption $C$ is a fraction of national income $Y$, i.e.,

$$C_i = c_i Y_i, \quad c_i \in [0, 1), i = 1, 2, 3$$
External trade

The imports of $i$ from $j$, denoted $M_{ji}$, is a fraction of national income of $i$, i.e.,

$$M_{ji} = m_{ji}Y_i, \quad m_{ji} \in [0,1], \quad i = 1, 2, 3$$

Total imports of $i$ is, therefore

$$M_i = m_{i1}Y_1 + m_{i2}Y_2 + m_{i3}Y_3$$

$$= m_iY_i$$

External trade

Note that in the above,

$$m_{i1} + m_{i2} + m_{i3} = m_i$$

We also impose the hypothesis that

$$0 \leq m_i < c_i < 1, \quad i = 1, 2, 3$$

since empirically, the marginal propensity to import is less than the marginal propensity to consume.

External trade

Observe that the exports of $j$ to $i$ are the imports of $i$ to $j$. We then have the following equations:

$$X_1 = M_{12} + M_{13}$$

$$= m_{12}Y_2 + m_{13}Y_3$$

$$X_2 = m_{21}Y_1 + m_{23}Y_3$$

$$X_3 = m_{31}Y_1 + m_{32}Y_2$$
External trade
The equilibrium condition for each country $i$ is

$$Y_i = C_i + I_i + G_i + X_i - M_i$$

where $G$ denotes government spending and $I$ denotes the level of investment. Take country 1: we have the equation

$$Y_1 = C_1 + I_1 + G_1 + X_1 - M_1$$

Note that

$$C_1 = c_1 Y_1, \quad M_1 = m_1 Y_1$$

$$X_1 = m_{12} Y_2 + m_{13} Y_3$$

Thus

$$Y_1 = (c_1 - m_1)Y_1$$

$$+ m_{12} Y_2 + m_{13} Y_3 + I_1 + G_1$$

Similarly, for country 2,

$$Y_2 = m_{21} Y_1 + (c_2 - m_2)Y_2$$

$$+ m_{23} Y_3 + I_2 + G_2$$

and for country 3,

$$Y_3 = m_{31} Y_1 + m_{32} Y_2$$

$$+ (c_3 - m_3)Y_3 + I_3 + G_3$$
External trade

In matrix notation,
\[
\begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{bmatrix} = \mathbf{A} \begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{bmatrix} + \begin{bmatrix}
l_1 + G_1 \\
l_2 + G_2 \\
l_3 + G_3
\end{bmatrix}
\]

where
\[
\mathbf{A} = \begin{bmatrix}
c_1 - m_1 & m_{12} & m_{13} \\
m_{21} & c_2 - m_2 & m_{23} \\
m_{31} & m_{32} & c_3 - m_3
\end{bmatrix}
\]

Let
\[
x = \begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{bmatrix}, \quad d = \begin{bmatrix}
l_1 + G_1 \\
l_2 + G_2 \\
l_3 + G_3
\end{bmatrix}
\]

Then
\[
x = \mathbf{A}x + d
\]

\[
x - \mathbf{A}x = d
\]

\[
(l - \mathbf{A})x = d
\]
Leontief model

**Definition.** An input-output model is an economic model which analyzes linkages among industries. The output of every industry is produced by using inputs from other industries, called intermediate inputs. Inputs outside these industries but used in production are called primary inputs.


Leontief model

**Definition.** Demand for outputs emanating from the production of more outputs among the industries are called intermediate demands. Output demand for final consumption by economic sectors outside these industries (e.g., households, government, external sector) are called final demands.

Leontief model

**Input-output table:**

<table>
<thead>
<tr>
<th>Sector 1</th>
<th>Sector 2</th>
<th>...</th>
<th>Sector n</th>
<th>Final demand</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector 1</td>
<td>x_{11}</td>
<td>...</td>
<td>x_{1n}</td>
<td>d_1</td>
<td>x_1</td>
</tr>
<tr>
<td>Sector 2</td>
<td>x_{21}</td>
<td>...</td>
<td>x_{2n}</td>
<td>d_2</td>
<td>x_2</td>
</tr>
<tr>
<td>Sector n</td>
<td>x_{n1}</td>
<td>...</td>
<td>x_{nn}</td>
<td>d_n</td>
<td>x_n</td>
</tr>
</tbody>
</table>

Primary input 1

<table>
<thead>
<tr>
<th>Sector 1</th>
<th>Sector 2</th>
<th>...</th>
<th>Sector n</th>
<th>Final demand</th>
<th>Total primary input</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary input 1</td>
<td>z_{11}</td>
<td>...</td>
<td>z_{1n}</td>
<td>y_1</td>
<td>z_1</td>
</tr>
<tr>
<td>Primary input 2</td>
<td>z_{21}</td>
<td>...</td>
<td>z_{2n}</td>
<td>y_2</td>
<td>z_2</td>
</tr>
</tbody>
</table>

Final input

<table>
<thead>
<tr>
<th>Sector 1</th>
<th>Sector 2</th>
<th>...</th>
<th>Sector n</th>
<th>Final input</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final input</td>
<td>x_1</td>
<td>...</td>
<td>x_n</td>
<td></td>
</tr>
</tbody>
</table>
Leontief model
where
\[ x_i = \text{output level of sector } i \]
\[ x_j = \text{output of sector } i \text{ supplied to sector } j \]
\[ d_i = \text{final demand of sector } i \]
\[ z_{kj} = \text{amount of primary input } k \text{ in sector } j \]

Leontief model
• Remark. An input-output table is also called a transactions table. Total output is also synonymous to total demand. It also assumes the following:
  i. each industry produces only one homogeneous output
  ii. each industry uses a fixed input ratio, and

Leontief model
  ii. production in every industry is subject to constant returns to scale.

These assumptions are called the economese conditions.
Leontief model

- **Definition.** In the input-output model, the matrix \( A \) defined below

\[
A = \begin{bmatrix} a_{ij} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} x_{ij} \\ x_j \\end{bmatrix}, \quad i,j=1,\ldots,n
\]

is called the technology matrix. Every \( a_{ij} \) is called a technical coefficient.

Leontief model

- **Remark.** Every technical coefficient from \( A \) gives the output of sector \( i \) supplied to sector \( j \) to produce one unit of output \( j \), i.e.,

\[
a_{ij} = \frac{x_{ij}}{x_j}
\]

Re-expressing the above, we get

\[
x_j = a_{ij}x_j
\]

Leontief model

If each sector produces output just enough to meet intermediate demands and final demand, we have, for a particular sector \( j \),

\[
x_j = a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n + d_j
\]

Doing this for all sectors \( j = 1,2,\ldots,n \), we have the system of equations
In matrix notation, we have

\[ x = Ax + d \]

or

\[ x - Ax = d \]
\[ (I - A)x = d \]

called the Leontief equation. The matrix \((I - A)\) is called the Leontief matrix. Suppose \((I - A)\) is nonsingular. Applying Theorem 4.16, we obtain a unique solution

\[ x^* = (I - A)^{-1} d \]

Note that the equation

\[ x^* = (I - A)^{-1} d \]

gives the gross output vector that must be produced in order to satisfy a given set of final demands.
**Example.** Consider an input-output table of Marlboro Country

<table>
<thead>
<tr>
<th>Sector 1</th>
<th>Sector 2</th>
<th>Sector 3</th>
<th>Final</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>472.3</td>
</tr>
<tr>
<td>Sector 1</td>
<td>44.7</td>
<td>214.5</td>
<td>15.8</td>
<td>197.3</td>
</tr>
<tr>
<td>Sector 2</td>
<td>42.8</td>
<td>210.1</td>
<td>174.2</td>
<td>527.1</td>
</tr>
<tr>
<td>Sector 3</td>
<td>28.6</td>
<td>149.9</td>
<td>290.8</td>
<td>668.3</td>
</tr>
<tr>
<td>Primary Input</td>
<td>356.2</td>
<td>714.5</td>
<td>1093.5</td>
<td></td>
</tr>
<tr>
<td>Total Input</td>
<td>472.3</td>
<td>1289.0</td>
<td>1574.3</td>
<td></td>
</tr>
</tbody>
</table>

We have the technology matrix

\[
A = \begin{bmatrix}
44.7 & 214.5 & 15.8 \\
472.3 & 1289.0 & 1574.3 \\
42.8 & 210.1 & 174.2 \\
472.3 & 1289.0 & 1574.3 \\
28.6 & 149.9 & 290.8 \\
472.3 & 1289.0 & 1574.3 \\
\end{bmatrix}
\]

Simplifying, we get

\[
A = \begin{bmatrix}
0.095 & 0.167 & 0.01 \\
0.091 & 0.163 & 0.111 \\
0.06 & 0.116 & 0.185 \\
\end{bmatrix}
\]

\[
(I - A) = \begin{bmatrix}
0.905 & -0.167 & -0.01 \\
-0.091 & 0.837 & -0.111 \\
-0.06 & -0.116 & 0.815 \\
\end{bmatrix}
\]
Exercise. Consider an input-output model with three sectors. Sector 1 is heavy industry, sector 2 is light industry, and sector 3 is agriculture. Suppose that the requirements are given by the following table:

<table>
<thead>
<tr>
<th>Industry</th>
<th>Heavy Industry</th>
<th>Light Industry</th>
<th>Agriculture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units of heavy</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>industry goods</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Units of light</td>
<td>0.3</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>industry goods</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Units of</td>
<td>0.2</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>agricultural</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>goods</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Suppose that final demands for the three goods are 85, 95, and 20 units, respectively.

Thus,

\[
(I - A)^{-1} = \begin{bmatrix}
1.131 & 0.232 & 0.045 \\
0.137 & 1.246 & 0.171 \\
0.103 & 0.194 & 1.255
\end{bmatrix}
\]

\[
x^* = (I - A)^{-1}d = \begin{bmatrix} 473.451 \\ 1290.000 \\ 1574.291 \end{bmatrix}
\]
Leontief model

Let \( x, y \) and \( z \) denote the number of units that have to be produced in the three sectors. Verify that the solution for the corresponding Leontief economy matrix equation is given by

\[
x^* = \begin{bmatrix} 150 & 200 & 100 \end{bmatrix}^T
\]

Leontief model

**Example.** Consider the following technology matrix

\[
A = \begin{bmatrix} 0.365 & 0 & 0 \\ 0 & 0.897 & 0 \\ 0 & 0 & 0.331 \end{bmatrix}
\]

What does this matrix tell about the three sectors in the economy?

Leontief model

**Example.** Consider the following technology matrix

\[
A = \begin{bmatrix} 0.05 & 0.1 & 0 \\ 0.1 & 0.25 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}
\]

What does this matrix tell about the three sectors in the economy?
Definition. A Leontief economy with technology matrix \( A \) is said to be viable if and only if the Leontief equation \((I - A)x = d\) has a nonnegative solution \( x^* \) for every nonnegative vector \( d \).

Definition. A square matrix \( C \) satisfies the Hawkins-Simon condition if and only if all its leading principal minors are positive.

Theorem 5.1. Let \( C \) be a square matrix such that its off-diagonal entries are nonpositive. Then the following are equivalent:

i. \( Cx = d \) has a solution \( x^* \geq \theta \) for some \( d > \theta \).

ii. \( Cx = d \) has a solution \( x^* > \theta \) for every \( d \geq \theta \).

iii. \( C \) satisfies the Hawkins-Simon condition.
Hawkins-Simon condition

**Example.** Consider again the input-output table of Marlboro Country with the technology matrix

\[
A = \begin{bmatrix}
0.095 & 0.167 & 0.01 \\
0.091 & 0.163 & 0.111 \\
0.06 & 0.116 & 0.185
\end{bmatrix}
\]

We verify Hawkins-Simon condition for \( I - A \):

\[
LPM_1 = \text{det} \begin{bmatrix} 0.905 \end{bmatrix} = 0.905 > 0
\]

\[
LPM_2 = \text{det} \begin{bmatrix} 0.905 & -0.167 \\
-0.091 & 0.837
\end{bmatrix} = 0.742 > 0
\]

\[
LPM_3 = \text{det}(I - A) = 0.592 > 0
\]

**Example.** Consider an input-output table of the economy of Hogwarts with two industries: Broomsticks and Magicwands.

<table>
<thead>
<tr>
<th></th>
<th>Broomsticks</th>
<th>Magicwands</th>
<th>Total demand</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broomsticks</td>
<td>189.0</td>
<td>239.5</td>
<td>615.5</td>
<td>1044.0</td>
</tr>
<tr>
<td>Magicwands</td>
<td>168.8</td>
<td>412.8</td>
<td>416.3</td>
<td>997.9</td>
</tr>
<tr>
<td>Primary input</td>
<td>686.2</td>
<td>345.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total input</td>
<td>1044.0</td>
<td>997.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Hawkins-Simon condition

i. Calculate the technology matrix.

ii. Obtain the Leontief matrix and verify that it satisfies Hawkins-Simon condition.

iii. Obtain the inverse of the Leontief matrix.

Hawkins-Simon condition

iv. If final demand for broomsticks decreased by 200 and final demand for magicwands increased by 50, obtain the new levels of output for the two sectors.

Hawkins-Simon condition

Solution. From the transactions table, we obtain the technology matrix

\[
A = \begin{bmatrix}
189 & 239.5 \\
1044 & 997.9 \\
168.8 & 412.8 \\
1044 & 997.9 \\
0.18 & 0.24 \\
0.16 & 0.41
\end{bmatrix}
\]
Hawkins-Simon condition

Thus, the Leontief matrix is given by

\[ \mathbf{L} = \mathbf{I} - \mathbf{A} \]

\[ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.18 & 0.24 \\ 0.16 & 0.41 \end{bmatrix} \]

\[ = \begin{bmatrix} 0.82 & -0.24 \\ -0.16 & 0.59 \end{bmatrix} \]

Hawkins-Simon condition

Obtaining the inverse, we use the formula for obtaining inverses of order 2 square matrices:

\[ (\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{0.44} \begin{bmatrix} 0.59 & 0.24 \\ 0.16 & 0.82 \end{bmatrix} \]

\[ = \begin{bmatrix} 1.34 & 0.55 \\ 0.36 & 1.86 \end{bmatrix} \]

Hawkins-Simon condition

**Example.** Consider the following transactions table of a certain country (in million pesos):

<table>
<thead>
<tr>
<th></th>
<th>Agriculture</th>
<th>Services</th>
<th>Final demand</th>
<th>Total demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agriculture</td>
<td>20</td>
<td>30</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>Services</td>
<td>40</td>
<td>40</td>
<td>20</td>
<td>100</td>
</tr>
</tbody>
</table>

chapter 5: input-output models
Hawkins-Simon condition

Reconstruct the transactions table with the same matrix of technical coefficients if total demands for agriculture and services are 200 and 500, respectively. Include the primary input requirements in the table.

Hawkins-Simon condition

**Exercise.** Recall the coefficient matrix in the model of international trade:

\[
A = \begin{bmatrix}
  c_1 - m_1 & m_{12} & m_{13}
  \\
  m_{21} & c_2 - m_2 & m_{23}
  \\
  m_{31} & m_{32} & c_3 - m_3
\end{bmatrix}
\]

Hawkins-Simon condition

By forming the corresponding Leontief matrix, use the Hawkins-Simon condition to show that the Leontief equation

\[
(I - A)x = d
\]

has a solution vector \(x^* \geq \theta\) for every \(d \geq \theta\).
Brauer-Solow condition

- **Remark.** In the Leontief equation \((I - A)x = d\), it may be laborious to calculate LPMs as the order of \((I - A)\) becomes sufficiently large. We then go to another condition which only involves row [or column] sums, called the Brauer-Solow condition.

Brauer-Solow condition

- **Theorem 5.2.** [Brauer-Solow] Let \(A\) be a nonnegative matrix. If at least one of the two conditions below is satisfied by \(A\), then \((I - A)\) satisfies the Hawkins-Simon condition:

\[
(i) \quad \text{row}_i = \sum_{j=1}^{n} a_{ij} < 1, \forall i = 1, 2, ..., n
\]

\[
(ii) \quad \text{col}_j = \sum_{i=1}^{n} a_{ij} < 1, \forall j = 1, 2, ..., n
\]
Brauer-Solow condition

Example. Consider again the technology matrix of Marlboro Country,

\[
A = \begin{bmatrix}
0.095 & 0.167 & 0.01 \\
0.091 & 0.163 & 0.111 \\
0.06 & 0.116 & 0.185
\end{bmatrix}
\]

Brauer-Solow condition

Remark. Brauer-Solow condition is a sufficient condition, i.e., not all matrices satisfying the Hawkins-Simon condition satisfies the Brauer-Solow condition. Example:

\[
A = \begin{bmatrix}
0.1 & 0.3 \\
2.0 & 0.2
\end{bmatrix}
\]

\[
I - A = \begin{bmatrix}
0.9 & -0.3 \\
-2.0 & 0.8
\end{bmatrix}
\]

Interpretation of \((I - A)^{-1}\)
Interpretation of \((I - A)^{-1}\)

- **Definition.** Let sector \(j\) increase in final demand by 1 unit while the other sectors remain the same. The \(j^{th}\) column sum
  \[
  L_j = \sum_{i=1}^{n} \ell_{ij}
  \]
  is called the **output multiplier** corresponding to a unit increase in the final demand of sector \(j\).

Interpretation of \((I - A)^{-1}\)

- **Example.** Consider
  \[
  (I - A)^{-1} = \begin{bmatrix}
  1.131 & 0.232 & 0.045 \\
  0.137 & 1.246 & 0.171 \\
  0.103 & 0.194 & 1.255
  \end{bmatrix}
  \]

Interpretation of \((I - A)^{-1}\)

- **Example.** Consider an input-output table of the economy of Hogwarts with two industries: Broomsticks and Magicwands.

<table>
<thead>
<tr>
<th></th>
<th>Broomsticks</th>
<th>Magicwands</th>
<th>Total demand</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broomsticks</td>
<td>189.0</td>
<td>239.5</td>
<td>615.5</td>
<td>1044.0</td>
</tr>
<tr>
<td>Magicwands</td>
<td>168.8</td>
<td>412.8</td>
<td>416.3</td>
<td>997.9</td>
</tr>
<tr>
<td>Primary input</td>
<td>686.2</td>
<td>345.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total input</td>
<td>1044.0</td>
<td>997.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Interpretation of \((I - A)^{-1}\)

- **Example.** The following inverse of a Leontief matrix was obtained from a transactions table whose entries are in pesos:

\[
\begin{bmatrix}
1.59 & 0.65 & 0.29 \\
0.68 & 1.50 & 0.39 \\
0.65 & 0.69 & 1.40
\end{bmatrix}
\]

Interpretation of \((I - A)^{-1}\)

1. Interpret the columns.
2. Derive the technology matrix \(A\) from which this matrix was obtained.

Further results on \((I - A)^{-1}\)
Further results on \((I - A)^{-1}\)

- **Theorem 5.3.** Let \(A\) be a nonnegative square matrix. Then the Leontief equation

\[
(I - A)x = d
\]

has a nonnegative solution \(x^*\) for every nonnegative vector \(d\) if and only if

\[
1 > \lambda = \max\{\alpha : \alpha \in \sigma(A)\}
\]

---

Further results on \((I - A)^{-1}\)

- **Theorem 5.4.** Let \(A\) be a technology matrix of a Leontief economy. Then the Leontief inverse matrix \(L^{-1} = (I - A)^{-1}\) satisfies the following properties

1. \((I - A)^{-1} \geq \Theta\)
2. \(\ell_i > 1, \ \forall i = 1, 2, \ldots, n\)

---

Further results on \((I - A)^{-1}\)

- **Example.** Recall the inverse of a Leontief matrix which was obtained from a transactions table whose entries are in pesos:

\[
(I - A)^{-1} = \begin{bmatrix}
1.59 & 0.65 & 0.29 \\
0.68 & 1.50 & 0.39 \\
0.65 & 0.69 & 1.40
\end{bmatrix}
\]
Demand for primary inputs

Example. Consider again the input-output table of Marlboro Country:

<table>
<thead>
<tr>
<th></th>
<th>Sector1</th>
<th>Sector2</th>
<th>Sector3</th>
<th>Final demand</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector1</td>
<td>44.7</td>
<td>214.5</td>
<td>15.8</td>
<td>197.3</td>
<td>472.3</td>
</tr>
<tr>
<td>Sector2</td>
<td>42.8</td>
<td>210.1</td>
<td>174.2</td>
<td>861.9</td>
<td>1289.0</td>
</tr>
<tr>
<td>Sector3</td>
<td>28.6</td>
<td>149.9</td>
<td>290.8</td>
<td>1105.0</td>
<td>1574.3</td>
</tr>
<tr>
<td>Primary inputs</td>
<td>356.2</td>
<td>714.5</td>
<td>1093.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total input</td>
<td>472.3</td>
<td>1289.0</td>
<td>1574.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Demand for primary inputs

Note that we obtained the corresponding technology matrix $A$:

$$A = \begin{bmatrix} 0.095 & 0.167 & 0.01 \\ 0.091 & 0.163 & 0.111 \\ 0.06 & 0.116 & 0.185 \end{bmatrix}$$

We do the same in the case of primary inputs, given in the next definition.
Demand for primary inputs

- **Definition.** In the input-output model, the matrix \( B \) defined by

\[
B = \begin{bmatrix}
  b_{kj} \\
  z_{kj} \\
  x_j
\end{bmatrix}
\]

is called the primary input matrix. Every \( b_{kj} \) is called a primary input coefficient.

Demand for primary inputs

- **Remark.** Every primary input coefficient from \( B \) gives the amount of primary input \( k \) needed to produce one unit of output in sector \( j \), i.e.,

\[
b_{kj} = \frac{z_{kj}}{x_j}
\]

Re-expressing the above, we get

\[
z_{kj} = b_{kj} x_j
\]

Demand for primary inputs

- **Input-output table:**

<table>
<thead>
<tr>
<th>Sector 1</th>
<th>Sector 2</th>
<th>Sector n</th>
<th>Total demand</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector 1</td>
<td>( x_{11} )</td>
<td>( x_{12} )</td>
<td>( \ldots )</td>
<td>( x_{1n} )</td>
</tr>
<tr>
<td>Sector 2</td>
<td>( x_{21} )</td>
<td>( x_{22} )</td>
<td>( \ldots )</td>
<td>( x_{2n} )</td>
</tr>
<tr>
<td>Sector n</td>
<td>( x_{n1} )</td>
<td>( x_{n2} )</td>
<td>( \ldots )</td>
<td>( x_{nn} )</td>
</tr>
</tbody>
</table>

| Primary input | \( z_{11} \) | \( z_{12} \) | \( \ldots \) | \( z_{1n} \) | \( y_1 \) |
| Primary input | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| Primary input | \( z_{n1} \) | \( z_{n2} \) | \( \ldots \) | \( z_{nn} \) | \( y_n \) |
| Total input | \( x_1 \) | \( x_2 \) | \( \ldots \) | \( x_n \) |
Demand for primary inputs

Note that the coefficients $b_{ij}$ are the primary input coefficients, and $b_{kj}x_j$ is the amount of primary input $k$ needed to produce $x_j$. Thus, the total amount of primary input $k$ in production is

$$y_k = b_{k1}x_1 + b_{k2}x_2 + \ldots + b_{kn}x_n$$

Doing this for all primary inputs $k = 1, 2, \ldots, m$, we have the system of equations

Demand for primary inputs

\[
\begin{align*}
y_1 &= b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n \\
y_2 &= b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n \\
&\vdots \\
y_m &= b_{m1}x_1 + b_{m2}x_2 + \ldots + b_{mn}x_n
\end{align*}
\]

In matrix notation, we have

$$y = Bx$$

Demand for primary inputs

Note that if $I - A$ is nonsingular, the Leontief equation

$$ (I - A)x = d $$

has a unique solution

$$ x^* = (I - A)^{-1}d $$

Thus,

$$ y^* = Bx^* $$

$$ y^* = B(I - A)^{-1}d $$
**Example.** Consider again the input-output table of Marlboro Country

<table>
<thead>
<tr>
<th>Country</th>
<th>Sector1</th>
<th>Sector2</th>
<th>Sector3</th>
<th>Final demand</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector1</td>
<td>44.7</td>
<td>214.5</td>
<td>15.8</td>
<td>197.3</td>
<td>472.3</td>
</tr>
<tr>
<td>Sector2</td>
<td>42.8</td>
<td>210.1</td>
<td>174.2</td>
<td>861.9</td>
<td>1289.0</td>
</tr>
<tr>
<td>Sector3</td>
<td>28.6</td>
<td>149.9</td>
<td>290.8</td>
<td>1105.0</td>
<td>1574.3</td>
</tr>
<tr>
<td>Primary input</td>
<td>356.2</td>
<td>714.5</td>
<td>1093.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total input</td>
<td>472.3</td>
<td>1289.0</td>
<td>1574.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Demand for primary inputs

We have the primary input matrix

\[
B = \begin{bmatrix} 356.2 & 714.5 & 1093.5 \\ 472.3 & 1289.0 & 1574.3 \end{bmatrix} = \begin{bmatrix} 0.754 & 0.554 & 0.695 \end{bmatrix}
\]

Suppose that the total amount of inputs available is 2250 units. Then, the total amount of inputs needed for production is

\[
y^* = Bx^*
\]

\[
= \begin{bmatrix} 0.754 \\ 0.554 \\ 0.695 \end{bmatrix} \begin{bmatrix} 473.451 \\ 1290.000 \\ 1574.291 \end{bmatrix}
= 2165.774
\]

Hence, there is enough inputs available to meet production requirements.
**Demand for primary inputs**

- **Exercise.** If there are 2000 units of input for Hogwarts, are the input requirements sufficient, given the transactions table below?

<table>
<thead>
<tr>
<th>Broomsticks</th>
<th>Magicwands</th>
<th>Final demand</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Broomsticks</td>
<td>189.0</td>
<td>239.5</td>
<td>615.5</td>
</tr>
<tr>
<td>Magicwands</td>
<td>168.8</td>
<td>412.8</td>
<td>416.3</td>
</tr>
<tr>
<td>Primary input</td>
<td>686.2</td>
<td>345.6</td>
<td></td>
</tr>
<tr>
<td>Total input</td>
<td>1044.0</td>
<td>997.9</td>
<td></td>
</tr>
</tbody>
</table>

**Duality**

- **Definition.** Consider the matrix equations (or equivalently, the systems of equations)

\[ Bx = b, \quad Cy = c \]

We say that these matrix equations are **dual** to each other iff

\[ B^T = C \]
Duality

- **Theorem 5.5.** Consider the dual matrix equations
  \[ Bx = b, \quad Cy = c \]
  If \( x^* \) and \( y^* \) are the respective solutions of the systems, then
  \[ b^Ty^* = c^Tx^* \]

- **Remark.** Consider the Leontief equation
  \[ (I - A)x = d \]  \[4\]
  A closed Leontief model
Closed model
- Recall the “usual” Leontief model and the economese conditions: these are embodied in the corresponding transactions table (or the input-output table) of a particular economy, industry, or system.

Closed model
- Input-output table:

<table>
<thead>
<tr>
<th>Sector 1</th>
<th>Sector 2</th>
<th>Sector n</th>
<th>Final demand</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector 1</td>
<td>x₁₁</td>
<td>x₁₂</td>
<td>...</td>
<td>x₁ₙ</td>
</tr>
<tr>
<td>Sector 2</td>
<td>x₂₁</td>
<td>x₂₂</td>
<td>...</td>
<td>x₂ₙ</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sector n</td>
<td>xₙ₁</td>
<td>xₙ₂</td>
<td>...</td>
<td>xₙₙ</td>
</tr>
</tbody>
</table>

Primary input 1
- z₁₁      z₁₂      ... z₁ₙ  y₁
- ...      ...      ...  ...

Primary input n
- zₙ₁      zₙ₂      ... zₙₙ  yₙ

Total input
- x₁      x₂      ... xₙ

Closed model
- Observe that final demand emanates from an external entity (say, households, government) and primary inputs originate from outside the producing sectors (or industries).
- For this reason, we call this “usual” input-output table an open Leontief model.

chapter 5: input-output models
Closed model

- Without loss of generality, consider a Leontief model with three sectors (1, 2, and 3).

<table>
<thead>
<tr>
<th>Sector 1</th>
<th>Sector 2</th>
<th>Sector 3</th>
<th>Final demand</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_{11}</td>
<td>x_{21}</td>
<td>x_{31}</td>
<td>d_1</td>
<td>y_1</td>
</tr>
<tr>
<td>x_{12}</td>
<td>x_{22}</td>
<td>x_{32}</td>
<td>d_2</td>
<td>y_2</td>
</tr>
<tr>
<td>x_{13}</td>
<td>x_{23}</td>
<td>x_{33}</td>
<td>d_3</td>
<td>y_3</td>
</tr>
</tbody>
</table>

Closed model

- Now suppose that final demand and primary inputs are within the model, say another sector (4).

<table>
<thead>
<tr>
<th>Sector 1</th>
<th>Sector 2</th>
<th>Sector 3</th>
<th>Sector 4</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_{11}</td>
<td>x_{21}</td>
<td>x_{31}</td>
<td>x_{41}</td>
<td>y_1</td>
</tr>
<tr>
<td>x_{12}</td>
<td>x_{22}</td>
<td>x_{32}</td>
<td>x_{42}</td>
<td>y_2</td>
</tr>
<tr>
<td>x_{13}</td>
<td>x_{23}</td>
<td>x_{33}</td>
<td>x_{43}</td>
<td>y_3</td>
</tr>
<tr>
<td>x_{14}</td>
<td>x_{24}</td>
<td>x_{34}</td>
<td>x_{44}</td>
<td>y_4</td>
</tr>
</tbody>
</table>

Closed model

- At first glance, this conversion would not seem to create any significant change in the analysis.
- However, **under the economese conditions, industries (or sectors) must follow a fixed-proportion ratio** of production of intermediate goods.

---

chapter 5: input-output models
Closed model

- Thus, primary inputs become intermediate inputs, and final demand follows some determined ratio (similar to that of the three producing sectors in the model).
- For this reason, this modification makes this Leontief model a closed Leontief model.

Closed model

- **Theorem 5.6.** A closed Leontief model

$$ (I - A)x = \theta $$

has a nonnegative solution $x^*$ if and only if $I - A$ is singular, i.e.,

$$ \det(I - A) = 0 $$

Productive matrices
Productive matrices

- **Definition.** A technology matrix $A$ is said to be **productive** iff given the Leontief matrix equation
  
  $$(I - A)x = d, \quad d = 1_n$$

  we have

  $$x > \Theta_n$$

Productive matrices

- **Example.** Determine which of the following matrices are productive:

  $$A_2 = \begin{bmatrix} 1/3 & 1/2 \\ 1/9 & 1/3 \end{bmatrix}$$

  $$A_3 = \begin{bmatrix} 1/2 & 1/4 & 1/10 \\ 1/4 & 1/4 & 2/5 \\ 1/10 & 2/5 & 1/4 \end{bmatrix}$$

Productive matrices

- **Theorem 5.7.** If a matrix $A$ is productive, then for every nonnegative vector $d$, the equation

  $$(I - A)x = d$$

  has a unique solution that satisfies

  $$x \geq d$$
Productive matrices

• **Theorem 5.8.** Let $A$ be a nonnegative matrix. Then the following are equivalent:
  
  (i) $A$ is productive.
  
  (ii) $\lim_{k \to \infty} A^k = \Theta$
  
  (iii) $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$, $A^0 = I$
  
  (iv) $(I - A)^{-1} \geq \Theta$

Productive matrices

• **Theorem 5.9.** [Perron-Frobenius]
  
  Let $A$ be a nonnegative matrix and define
  
  $\lambda_A = \inf \left\{ \mu > 0 : \frac{1}{\mu} A \text{ productive} \right\}$

  Then $\lambda_A \geq 0$ is the largest [real] eigenvalue of $A$ and has an associated eigenvector $x \geq \Theta$

  (note: “inf” = infimum).

Productive matrices

• **Definition.** Let $A$ be a nonnegative matrix and define $\lambda_A$ as in Theorem 5.9. We call $\lambda_A$ the Perron-Frobenius root of $A$. 

---

chapter 5: input-output models
Exercise. Find the Perron-Frobenius root and an associated eigenvector for each of the following matrices:

\[ A = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{9} & \frac{1}{3} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \]

To end...

We could of course, dismiss the rigorous proof as being superfluous: if a theorem is geometrically obvious why prove it? This was exactly the attitude taken in the eighteenth century. The result, in the nineteenth century, was chaos and confusion: for intuition, unsupported by logic, habitually assumes that everything is much nicer behaved than it really is.

I. Stewart [1975]